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EMPIRICAL FORMULÆ FOR APPROXIMATE COMPUTATION.

By the Late ANSEL N. KELLOGG, of New York City.

[NOTE. The following paper is here printed in the exact form in which it was left by the author at the time of his death, except only a few necessary verbal alterations which are distinguished from the other parts of the paper by an enclosure of square brackets.]

The following symbols have the same signification, throughout these formulæ, the only exception being when the letter “*d*” is used with *a*, *b*, *c*, etc.

$n = \frac{1}{m}$ = index of root.

$m = \frac{1}{n}$ = index of power.

q = quantity, or $\frac{q}{p}$, if fractional.

s = sum of $q + p$ (or the semi-sum).*

d = difference of q and p (or the demi-difference).*

$t = \frac{n^2 - 1}{3}$.

U = the sub-square, or under-square = $(\sqrt{q} - 1)^2$.

U_4 = the under-fourth = $(\sqrt[4]{q} - 1)^4$, etc.

E = Napierian logarithm of number.

K = logarithm of number to base 4.

r = root of number.

*These last values for s and d cannot be used in the same equation with p and q .

$\approx$ signifies nearly equal.

$\approx\approx$ signifies very nearly equal.

Mercator's formula for the extraction of roots of numbers near unity was equivalent to my own formula No. 1, as shown in Hutton's *Tracts on Mathematics*, Vol. I.

$$\frac{ns+d}{ns-d} \approx \sqrt[n]{q}, \text{ nearly } \dots \dots \dots (1).$$

Hutton says he gave a formula for the correction of this result, but I have never been able to find it. Hutton himself gives the derivation of the above formula.

First correction of above :

$$\sqrt[n]{q} \approx \frac{ns+d - \frac{(n^2-1)d^2}{3ns}}{ns-d - \frac{(n^2-1)d^2}{3ns}}$$

Substituting t for $\frac{n^2-1}{3}$, we have

$$\sqrt[n]{q} \approx \frac{ns+d - \frac{td^2}{ns}}{ns-d - \frac{td^2}{ns}} \dots \dots \dots (2).$$

Second correction of above :

$$\sqrt[n]{\frac{q}{p}} \approx \frac{ns+d - \frac{t(d^2 + [\sqrt{q} - \sqrt{p}]^4)}{ns}}{ns-d - \frac{t(d^2 + [\sqrt{q} - \sqrt{p}]^4)}{ns}}; \dots \dots \dots (3).$$

or by reducing,

$$\sqrt[n]{q} \approx \frac{(n^2+2)s + (2n^2-2)\sqrt{s^2-d^2} + 3nd}{(n^2+2)s + (2n^2-2)\sqrt{s^2-d^2} - 3nd} \dots \dots \dots (4).$$

This nearly equals

$$\sqrt[n]{q} \approx \frac{ns+d - \frac{n^2-1}{3} \times \frac{d^2}{s^2 - \frac{d^2}{4s}}}{ns-d - \frac{n^2-1}{3} \times \frac{d^2}{s^2 - \frac{d^2}{4s}}} \dots \dots \dots (5).$$

Simpler than these, and nearly related to (3) is

$${}^n\sqrt{q} = \frac{ns + d - \frac{td^2}{ns - \frac{td^2}{ns}}}{ns - d - \frac{td^2}{ns - \frac{td^2}{ns}}} \dots\dots\dots (6).$$

Very simple and excellent is

$${}^n\sqrt{q} = \frac{ns + d - \frac{2tU}{n}}{ns - d - \frac{2tU}{n}} \dots\dots\dots (7).$$

All the foregoing are what I call *equidistant* processes, because for all values of n , the difference between the numerator and denominator of the result is $2d$, (or some multiple); that is, the subtractive corrections following d are the same in both terms, whether q/p be a proper or an improper fraction.

Of the same nature, but differently derived, is formula (8) and its equivalent (8½).

$${}^n\sqrt{q} = \frac{n(2s^2 - d^2) + d^2 + 2ds}{n(2s^2 - d^2) + d^2 - 2ds} \dots\dots\dots (8).$$

$${}^n\sqrt{q} = \frac{n(2s^2 - d^2) + d(2s + d)}{n(2s^2 - d^2) - d(2s - d)} \dots\dots\dots (8\frac{1}{2}).$$

The fact that all these fractional formulae are symmetrical makes their operation comparatively simple.

In extracting roots of high numbers

$${}^n\sqrt{q} = 2^{\frac{m}{n}} \times \sqrt[n]{\frac{q}{2^m}},$$

and we are thus always enabled to use $q/2^m$ between the limits $\frac{1}{2}$ and 2. Hence, the following equation becomes of value :

$${}^{n/m}\sqrt{2} = \frac{3n + m - \frac{n^2 - m^2}{9n}}{3n - m - \frac{n^2 - m^2}{9n}}, \dots\dots\dots (9),$$

$$\text{or } \sqrt[n]{2} = \frac{3n+m-\frac{n^2-m^2}{9n-\frac{n^2-m^2}{3n}}}{3n-m-\frac{n^2-m^2}{9n-\frac{n^2-m^2}{3n}}}, \dots\dots\dots (10),$$

$$\text{or } \sqrt[n]{2} = \frac{3n+m-\frac{29(n^2-m^2)}{507n}}{3n-m-\frac{29(n^2-m^2)}{507n}}, \dots\dots\dots (11).$$

In a latent form, the equidistant principle is also present in the following :

$$\sqrt[n]{\frac{q}{p}} = \frac{(2n^2+3n+1)q^2+(8n^2-2)qp+(2n^2-3n+1)p^2}{(2n^2-3n+1)q^2+(8n^2-2)qp+(2n^2+3n+1)p^2} \dots\dots\dots (13).$$

In some of the following applications p is taken $=1$.

This [viz. (13)] becomes :

$$\text{for } \sqrt[2]{q} \dots\dots\dots \frac{5q^2+10q+1}{q^2+10q+5} \dots\dots\dots (13-2)$$

$$\text{for } \sqrt[3]{q} \dots\dots\dots \frac{14q^2+35q+5}{5q^2+35q+14} \dots\dots\dots (13-3)$$

$$\text{for } \sqrt[4]{\frac{q}{p}} \dots\dots\dots \frac{15q^2+42qp+7p^2}{7q^2+42qp+15p^2} \dots\dots\dots (13-4)$$

$$\text{for } \sqrt[5]{q} \dots\dots\dots \frac{11q^2+33q+6}{6q^2+33q+11} \dots\dots\dots (13-5)$$

$$\text{for } \sqrt[6]{q} \dots\dots\dots \frac{91q^2+286q+55}{55q^2+286q+91} \dots\dots\dots (13-6)$$

$$\text{for } \sqrt[7]{\frac{q}{p}} \dots\dots\dots \frac{20q^2+65qp+13}{13q^2+65qp+20} \dots\dots\dots (13-7)$$

$$\text{for } \sqrt[8]{q} \dots\dots\dots \frac{51q^2+170q+35}{35q^2+170q+51} \dots\dots\dots (13-8)$$

$$\text{for } \sqrt[9]{q} \dots\dots\dots \frac{95q^2+323q+68}{68q^2+323q+95} \dots\dots\dots (13-9)$$

$$\text{for } {}^{10}\sqrt{\frac{q}{p}} \dots\dots\dots \frac{77q^2 + 266qp + 57p^2}{57q^2 + 266qp + 77p^2} \dots\dots\dots (13-10).$$

$$\text{for } {}^{100}\sqrt{q} \dots\dots\dots \frac{6767q^2 + 26666q + 6567}{6567q^2 + 26666q + 6767} \dots\dots\dots (13-100).$$

For very high indices use, without sensible error,

$${}^n\sqrt{\frac{q}{p}} = \frac{(2n+3)q^2 + 8npq + (2n-3)p^2}{(2n-3)q^2 + 8npq + (2n+3)p^2} \dots\dots\dots (14).$$

(14) is the equivalent of (2).

Upon the logarithmic function depend the following formulæ :

$${}^n\sqrt{q} = \frac{n + \frac{E}{2} + \left(\frac{E}{6} \times \frac{r-1}{r+1}\right)}{n - \frac{E}{2} + \left(\frac{E}{6} \times \frac{r-1}{r+1}\right)} \dots\dots\dots (15).$$

$${}^n\sqrt{q} = \frac{3 + \frac{2E}{n} + \frac{E^2}{2n^2}}{3 - \frac{E}{n}} \dots\dots\dots (16).$$

$$E, \text{ the logarithm of } q = \frac{3(q^2-1)}{(q+1)^2 + 2q} \dots\dots\dots (17),$$

$$\text{or, if } q \text{ be fractional, } = \frac{3(q^2-p^2)}{(q+p)^2 + 2qp} \dots\dots\dots (18),$$

$$\text{or, in terms of } d \text{ and } s = \frac{6ds}{3s^2-d^2} \dots\dots\dots (19).$$

This value of E , if q be between .9 and 1.1, is true to the seventh decimal, but may be corrected with very great accuracy, even up to the ninth or tenth decimal, by adding to the result the $\frac{4d^2}{45s^2}$ th part of itself.

If, however q be so great as

- 1.7 or so small as .6 use 44 in place of 45
- 1.8 or so small as .56 use 43 in place of 45
- 1.9 or so small as .53 use 42 in place of 45
- 2. or so small as .5 use 41 in place of 45.

The accuracy of these formulæ will appear from the natural logarithms in the next [paragraph].

[Some verifications of formula (18) :

For $q/p = \frac{11}{10} : -$

By the formula : $\log(q/p) = \frac{63}{661} = .0953101,$
error = 1 in last place.

By the tables : 2.3978951
2.3025851

$\log(q/p) = .0953100$

For $q/p = \frac{11}{10} : -$

By the formula : $\log(q/p) = \frac{123}{1441} = .0645385,$
correct to last place.

By the tables : 2.7725887
2.7080502
0.0645385

For $q/p = \frac{11}{10} : -$

By the formula : $\log(q/p) = \frac{883}{8861} = .00995033.]$

Reconverting this by (15) we have

$$\frac{1 + .004975165 + .000008251}{1 - .004975165 + .000008251} = \frac{1.004983416}{.995033086}.$$

To this denominator add the 100th part,

$$\begin{array}{r} .995033086 \\ .009950331 \\ \hline 1.004983417 \end{array}$$

Hence the real number is $\frac{11}{10}$.

From the foregoing we have another value of ${}^n_1/q$ as follows :

$${}^n_1/q = \frac{2n + \left(3 + \frac{r-1}{r+1}\right) \left(\frac{q^2 - p^2}{q^2 + 4qp + p^2}\right)}{2n - \left(3 - \frac{r-1}{r+1}\right) \left(\frac{q^2 - p^2}{q^2 + 4qp + p^2}\right)} \dots \dots \dots (20).$$

(Of course r can only be taken crudely, but may by successive steps, until the required approximation is reached.)

Another formula akin to (15), for values of q (or r) in terms of E is

$${}^n_1/q = \frac{n + \frac{E}{2} + \frac{E^2}{12}}{n - \frac{E}{2} + \frac{E^2}{12}} \dots \dots \dots (15\frac{1}{2}).$$

Let us call c , which is equal to $\frac{q-1}{E}$, a *root centre*, meaning thereby that,

for all values of q , and degrees of n , $\frac{c + \frac{q-1}{q+1}}{c - \frac{q-1}{q+1}}$ nearly equals ${}^n\sqrt{q}$. Then

$$\frac{nc + \frac{q-1}{2} + \frac{(q-1)(r-1)}{6(r+1)}}{nc - \frac{q-1}{2} + \frac{(q-1)(r-1)}{6(r+1)}} = {}^n\sqrt{q} \dots\dots\dots (21).$$

c , if q is under 10, is nearly $\frac{2\sqrt{q} + \frac{q+1}{2}}{3} \dots\dots\dots (22).$

c , if q is between 50 and 250, is nearly same — $\frac{q^3}{500} \dots\dots\dots (23).$

c , if q is between 10 and 50 [correction not given].

FACTOR PROCESS.

Take $\frac{b}{a} \times \frac{c}{b} \times \dots\dots\dots \times \frac{h}{g} = q$; then $\frac{h}{a} = q$ and the series is consecutive.

Take $\frac{b}{a} \times \frac{d}{c} \times \dots\dots\dots \times \frac{h}{g} = q$; then $\frac{bd\dots h}{ae\dots g} = q$.

Now if, to take the n th root of q , we assume n terms, consecutive or non-consecutive, and nearly equal in value

$${}^n\sqrt{q} = \frac{\frac{\sqrt{b}}{\sqrt{a}} + \frac{\sqrt{d}}{\sqrt{c}} + \dots\dots\dots + \frac{\sqrt{h}}{\sqrt{g}}}{\frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{c}}{\sqrt{d}} + \dots\dots\dots + \frac{\sqrt{g}}{\sqrt{h}}}, \dots\dots\dots (24),$$

wherein the numerators are the square roots of the terms and the denominators the reciprocals thereof.

If the terms are consecutive and odd in number

$${}^n\sqrt{q} = \frac{ah + 2bg + \dots\dots\dots + 2de}{2ag + 2bf + \dots\dots\dots d^2}; \dots\dots\dots (25)$$

but if consecutive and even in number

$${}^n\sqrt{q} = \frac{ai + 2bh + \dots + e^2}{2ah + 2bg + \dots + 2de} \dots \dots \dots (25\frac{1}{2}).$$

Also if the series is consecutive,

$${}^n\sqrt{q} = \frac{(g+h)b + (f+g)c + \dots + (b+c)g + (a+b)h}{(g+h)a + (f+g)b + \dots + (b+c)f + (a+b)h}.$$

(25) and (25½) are called the *diagonal* process.

To separate a quantity q/p into consecutive factors which shall be nearly equal, and which shall be as many as there are units in the index of the root to be extracted, and whose differences shall also be in arithmetical progression, use for expansor $\frac{n^2 s}{d^2}$, and then arrange terms with differences themselves differing by unity. This may be done by dividing d' , the difference of the expanded fraction q'/p' , or $q' - p'$, by n , and making the final interval $\frac{n-1}{2}$ less than the quotient, —after which ascend accordingly.

Illustrations: 1. Separate $\frac{27}{13.5}$ into three factors of above nature:—

$$\frac{n^2 s}{d^2} = 13.5. \quad \text{Then } \frac{q'}{p'} = \frac{27}{13.5}, \quad \frac{13.5}{3} = 4.5, \quad \frac{n-1}{2} = 1.$$

Then first interval = 3.5. Therefore :

$$\frac{17}{13.5}, \quad \frac{21.5}{17}, \quad \frac{27}{21.5}, \text{ or } \frac{34}{27}, \quad \frac{43}{34}, \quad \frac{54}{43}$$

are the desired terms.

Applying the diagonal process :

$$\frac{1458 + 1462 + 1462}{1161 + 1161 + 1156} = \frac{4382}{3478} = \frac{2191}{1739} = 1.2599195. \quad \text{Error} = .0000015.$$

2. Separate $\frac{5}{8}$ into 4 factors:—

Expansor = 16 gives $\frac{5}{8} = \frac{8.9}{8}$; first interval = 6.5.

[The series of consecutive factors is]

$$\frac{54.5}{48}, \quad \frac{62}{54.5}, \quad \frac{70.5}{62}, \quad \frac{80}{70.5}, \text{ or } \frac{109}{96}, \quad \frac{124}{109}, \quad \frac{141}{124}, \quad \frac{160}{141}.$$

3. Separate $\frac{3}{4}$ into five factors:—

Expansor = 12.5; first interval = 3, [and the series of consecutive factors is]

$$\frac{15.5}{12.5}, \quad \frac{19.5}{15.5}, \quad \frac{24.5}{19.5}, \quad \frac{30.5}{24.5}, \quad \frac{37.5}{30.5}, \text{ or } \frac{31}{25}, \quad \frac{39}{31}, \quad \frac{49}{39}, \quad \frac{61}{49}, \quad \frac{75}{61}.$$

If the factors are consecutive and in arithmetical progression, then

$${}^n\sqrt{q} = \frac{n^2 q + \frac{n^2 - 1}{6}(q-1)^2}{n^2 q - n(q-1) + \frac{(n-1)(n-2)}{6}(q-1)^2} \dots\dots\dots (27).$$

All quantities p/q may be represented in n terms, which group in three classes as follows :

$$\left(\frac{b}{a}\right)^j \times \left(\frac{d}{c}\right)^k \times \left(\frac{b}{e}\right)^l = q/p, \text{ where } j+k+l=n.$$

$$\text{Then } {}^n\sqrt{q} = \frac{j(fc+de)b + k(fa+be)d + l(ad+bc)f}{j(fc+de)a + k(fa+be)c + l(ad+bc)e} \dots\dots\dots (28).$$

The above is called the *three-class* process.

Sometimes a quantity will reasonably resolve into n terms, which group in only two classes. Hence the following *two-class* process :

$$\left(\frac{b}{a}\right)^j \times \left(\frac{d}{c}\right)^k = q;$$

$${}^n\sqrt{q} = \frac{j(ad+bc+2cd)b + k(ad+bc+2ab)d}{j(ad+bc+2cd)a + k(ad+bc+2ab)c} \dots\dots\dots (29).$$

Or, the following, simpler but not so good :

$${}^n\sqrt{q} = \frac{(c+d)jb + (a+b)kd}{(c+d)ja + (a+b)kc} \dots\dots\dots (30).$$

Of all these processes the three-class (28) is the most trust-worthy. When q does not naturally resolve in such terms, take q/pv which does, and extract ${}^n_1\sqrt{v}$, and multiply results.

In the consecutive series

$$\frac{b}{a} \times \frac{c}{b} \times \dots\dots\dots \times \frac{e}{d} \times \frac{b}{e} = q,$$

take a new term, of the first expanded q times, that is qb/qa . Then drop the term b/a , and call $qb=q$ and $qa=f$, giving the equation

$$\frac{c}{b} \times \frac{d}{c} \times \dots \times \frac{f}{e} \times \frac{g}{f} = q.$$

Now apply the diagonal process, according to the spirit, and not the letter, of (25), and we have,

$$\frac{bg+2cf+2de}{2bf+2ce+d^2} = \sqrt[n]{q} \dots \dots \dots (25\frac{1}{2}).$$

Now this result will be found no nearer than the result in (25) but the mean of the two will give a close result. It is not, however, a formula of value, and I think there are cases where the error of (25) and (25½) are both on the *same* side. Hence, their mean would be of no value in particular.

PROCESS FOR SPECIAL ROOTS.

For Square Root: [Hutton gives $\frac{ac+b^2}{2ab} = \sqrt{\frac{b}{a} \times \frac{c}{b}}$]

$$\sqrt[2]{q} = \frac{(c+d)b + (a+b)d}{(c+d)a + (a+b)c} \dots \dots \dots (31).$$

$$\sqrt[2]{q} = \frac{3bcd + 3abd + b^2c + d^2a}{3abc + 3acd + a^2d + b^2c} \dots \dots \dots (32).$$

$$\sqrt[2]{q} = \frac{(\sqrt{cd} \times b) + (\sqrt{ab} \cdot d)}{(\sqrt{cd} \times a) + (\sqrt{ab} \times c)},$$

$$\text{or} = \frac{b\sqrt{cd} + d\sqrt{ab}}{a\sqrt{cd} + c\sqrt{ab}} \dots \dots \dots (33).$$

If we call r an approximate value of the required square root, then

$$\frac{q+r}{p+r} = \sqrt{\frac{q}{p}} \dots \dots \dots (34).$$

$$\text{or} \frac{q^2 + 6r^2q + r^4}{4r(q+r^2)} = \sqrt[2]{q} \dots \dots \dots (35).$$

For Cube Roots:—

$$\frac{7q^3 + 42q^2 + 30q + 2}{2q^3 + 30q^2 + 42q + 7} = \sqrt[3]{q} \dots \dots \dots (36).$$

Also, if $\frac{b}{a} \times \frac{d}{c} \times \frac{f}{e} = q$,

$${}^3\sqrt{q} = \frac{bdc + bcf + adf}{acf + ade + bce} \dots \dots \dots (37).$$

This is the best of all cube root processes, and I call it the *interweaving* process. It has remarkable properties.

For the $\frac{3}{2}$ Root :—If $\frac{b}{a} \times \frac{c}{b} \times \frac{d}{c} = q$; then

$$\frac{2bd + c^2}{2ac + b^2} = {}^{\frac{3}{2}}\sqrt{q}, \text{ or } {}^3\sqrt{q^2} \dots \dots \dots (38).$$

For the Fourth Root :—Let

$$\frac{b}{a} \times \frac{c}{b} \times \frac{d}{c} \times \frac{e}{d} = q.$$

$$\text{Then } {}^4\sqrt{q} = \frac{(be + cd)b + (ae + bd)e + (ad + bc)d + (ac + b^2)e}{(be + cd)a + (ae + bd)b + (ad + bc)c + (ac + b^2)d} \dots \dots \dots (39).$$

UNDER-SQUARE FORMULE.

$$U = (\sqrt{q} - \sqrt{p})^2, U_4 = (\sqrt[4]{q} - \sqrt[4]{p})^4, U_8 = (\sqrt[8]{q} - \sqrt[8]{p})^8 \dots \dots \dots$$

(7) is an under-square formula and, if q is an even square, is precisely equivalent to the $\frac{1}{2}n$ th root of $\sqrt{q}$, by the first correction of formula 1.

The first correction of (7) is

$${}^n\sqrt{\frac{q}{p}} = \frac{ns + d - \frac{2(n^2 - 1)}{3n}U - \frac{n^2 - 4}{6n}U_4}{ns - d - \frac{2(n^2 - 1)}{n}U - \frac{n^2 - 4}{6n}U_4} \dots \dots \dots (40).$$

$$\text{Also, } {}^n\sqrt{\frac{q}{p}} = \frac{ns + d - \frac{2(n^2 - 1)}{3n}U - \frac{(E - 1)U}{72} \times \frac{n(n - 2)}{n - 1}}{ns - d - \frac{2(n^2 - 1)}{3n}U - \frac{(E - 1)U}{72} \times \frac{n(n - 2)}{n - 1}} \dots \dots \dots (41).$$

Still another but complex value of ${}^n\sqrt{\frac{q}{p}}$ is :

$$\frac{ns + d - \frac{(2n^2 - n)}{3n}U - \frac{(n^2 - 4)}{6n}U_4 - \frac{(n^2 - 16)({}^8\sqrt{q} - {}^8\sqrt{p})^4 \times (\sqrt{q} + 1)\sqrt{p}}{*12n}}{ns - d - \frac{(2n^2 - n)}{3n}U - \frac{(n^2 - 4)}{6n}U_4 - \frac{(n^2 - 16)({}^8\sqrt{q} - {}^8\sqrt{p})^4 \times (\sqrt{q} + 1)\sqrt{p}}{*12n}} \dots \dots (42).$$

*It seems that $11n$ is better in actual practice.

For cube root use for coefficients $\frac{1}{9}^6$ and $\frac{5}{18}$ instead of $\frac{1}{9}^6$ and $\frac{5}{18}$.

Illustrations: All the equidistant formulæ, except these, approach accuracy only when q is near unity. No such restriction binds these [the under-square] formulæ, especially those which are most developed. And here, let me say, is undoubtedly the beginning of a series which I think the Calculi would unfold, and I trust some friend of science will take the burden of solving it. So developed, I am satisfied that all roots of all numbers would be extractable to any required degree of accuracy.

1. Taking (40) extract $\sqrt[3]{4096}$.

$$\text{Root} = \frac{3 \times 4097 + 4095 - \frac{1}{9}^6 \times 3969 - \frac{5}{18} \times 2401}{3 \times 4097 - 4095 - \frac{1}{9}^6 \times 3969 - \frac{5}{18} \times 2401}$$

$$= \frac{12291 + 4095 - 7056 - 667 \text{ nearly}}{12291 - 4095 - 7056 - 667 \text{ nearly}} = \frac{8^6 6^3}{4^7 3} = 18 +;$$

but taking $\frac{5}{18}$ instead of $\frac{5}{18} = \frac{8^6 9^6}{5^6 6^3} = 17 +.$

2. Taking $\sqrt[4]{4096}$, we have

$$\text{Root} = \frac{4 \times 4097 + 4095 - \frac{5}{2} \times 3969 - \frac{1}{2} \times 2401}{4 \times 4097 - 4095 - \frac{5}{2} \times 3969 - \frac{1}{2} \times 2401} = \frac{9^3 6^9}{1^1 1^1 0} = 8.$$

3. Take $\sqrt[6]{4096}$. By (42),

$$\text{Root} = \frac{6 \times 4097 + 4095 - \frac{3}{9}^5 \times 3969 - \frac{8}{9} \times 2401 - \frac{5}{18} \times \frac{1}{9}^0 \times 65}{6 \times 4097 - 4095 - \frac{3}{9}^5 \times 3969 - \frac{8}{9} \times 2401 - \frac{5}{18} \times \frac{1}{9}^0 \times 65} = \frac{11907.2}{3717} = 3.2,$$

but should = 4.

TO SUM A SERIES: $S=1, r, r^2, \dots, r^{n-1}, n \text{ terms.}$

Let $s=\text{ratio}+1=r+1, d=r-1.$

$$S = 1 + \frac{2n}{s - nd + \frac{(n^2 - 1)d^2}{3s}} \dots \dots \dots (43).$$

If s exceeds 2, this is not accurate enough to be of value.

CUBE ROOT BY DIFFERENCE METHOD.

Take $a^3 < q, b^3 > q.$ Call $q - a^3 = A, b^3 - q = B.$ Then

$$\frac{a^2 B + b^2 A}{aB + bA} = \sqrt[3]{q} \dots \dots \dots (44).$$

CONVENIENT FORMULA FOR ROOTS OF 2.

$${}^n\sqrt{2} = \frac{101n + 35 + \frac{35(r-1)}{6(r+1)}}{101n - 35 + \frac{35(r-1)}{6(r+1)}} \dots\dots\dots (45).$$

A ROUGH VALUE OF q IN TERMS OF E .

$$q = \frac{\sqrt{9 + 3E^2} + 2E}{3 - E} \dots\dots\dots (46).$$

LOGARITHMS.

A curious fact, but scarcely a useful one, is : The logarithm of any number approaches

$$K_B = \frac{\sqrt{B} \times (q - B^m)}{\sqrt{B} \times (\sqrt{B} - 1)B^m + (\sqrt{B} - 1)q} + m, \dots\dots\dots (47),$$

wherein K is the logarithm, B the base of the system, m the characteristic of the logarithm of q , the quantity. Now if $B=4$, the formula becomes

$$K_4 = \frac{2(q - B^m)}{2B^m + q} + m, \dots\dots\dots (48).$$

$$\text{If } q = q/p, \text{ then } K_4 = \frac{2q - 2B^m p}{2B^m p + q} + m \dots\dots\dots (49).$$

$$\text{If } K_4 = \frac{m+c}{d}, \text{ then } q = B^m \left(1 + \frac{3c}{2d-c}\right) \dots\dots\dots (50)$$

$$= B^m \frac{2d+2c}{2d-c} \dots\dots\dots (51).$$

Since Napierian $\log E = \frac{4}{11} K$, and also, since

$${}^n\sqrt{q} = \frac{\frac{44}{61}K + \frac{1}{2n}}{\frac{44}{61}K - \frac{1}{2n}}$$

roughly, then

$${}_n\sqrt{q} = \frac{\frac{44}{61}K + \frac{1}{2n}}{\frac{44}{61}K - \frac{1}{2n}} \text{ roughly } \dots\dots\dots (52).$$

The term K also may be understood to mean the logarithm of q to base 4.

$$K \text{ of } 10 = \frac{5}{3}; \text{ then } q = 4(1 + \frac{5}{4}) = 10.$$

The error by this method is easily represented by a curve.

In the fractional processes continually occur the coaddition of fractions, or the adding of numerators together, and of denominators also, when the latter are not the same. When the fractions, two or more, are "embracing," the results are close, and if "harmoniously embracing" then positively accurate geometric averages. They are harmonious when the product of the two terms of each fraction are the same.

Thus, $\frac{8}{6}$ and $\frac{7}{8}$ are embracing, and their co-sum doubled is $2 \times \frac{15}{11} = 2.727 +$. The real sum is $\frac{8}{6} + \frac{7}{8} = 2.767 +$. $\frac{1^2}{4}$, $\frac{1^1}{8}$ and $\frac{1^0}{9}$ are embracing, and their co-sum tripled is $3 \times \frac{3}{4} = 4.125$. The real sum is $\frac{2^1 1^1 7}{6^1 6^1 4^1} = 4.200$. $\frac{2}{4}$ and $\frac{6}{8}$ are harmoniously embracing. Their co-sum is $\frac{1}{2}$ or $\frac{3}{2}$ which is their square root and precise geometric average.

Rough addition may be performed by co-addition. Thus, $\frac{1^0}{7} + \frac{2^1 1^1}{3^1 3^1} = \frac{2^1 1^1}{3^1 3^1}$, or $\frac{1}{11}$, which doubled differs from $\frac{3^1 4^1}{4^1 4^1}$ as 3817 differs from 3808, or 1 in 423 parts.

In the foregoing processes for extraction of roots, calculation of logarithms, etc., excepting the factor-process, the accuracy increases rapidly as q approaches unity. In general they have value only when q lies between .5 and 2. Their value should be tested by logarithms. When q exceeds 2, it may generally be divided by the n th power of some simple quantity, which will bring it near unity. If this be difficult, use the nearest power of 2 (say m) as previously explained.

Many apparently absurd problems are readily answered by these formulæ. Thus : *If the cube root of 2.5 = $\frac{1}{4}$ what is the 4th root of 3.333 + ?*

Using the 2-class process, formula (29),—

$$\left(\frac{19}{14}\right)^3 \times \left(\frac{4}{3}\right) = (\text{say to } 3\frac{1}{3}) \frac{(125 \times 3 \times 19) + (645 \times 4)}{(375 \times 14) + (645 \times 3)}$$

$$= 3\frac{795}{1186} = 3\frac{124}{1432} = 1.35545, [\text{error here}]$$

or by (30),

$$\frac{399 + 132}{294 + 99} = \frac{531}{393} = \frac{177}{131} = 1.35114.$$

True answer = 1.35120.

COMPARISON OF FRACTION.

It is frequently necessary, or desirable to compare the values of two unwieldily, yet not very unequal fractions, or to ascertain approximately the comparative value of two ordinary fractions quickly, even if only approximately. Hence the following :

Compare b/a with d/c . Suppose them nearly equal to z/y , and divide each fraction by it, giving by/az and dy/cz . Subtract unity from each, and we have $\frac{by \pm az}{az}$ which make equal to $\frac{1}{uaz}$, and $\frac{dy - cz}{cz}$ which make equal to $\frac{1}{vcz}$. Then

$$\frac{uaz \mp vcz}{(uaz) \times (vcz)} = \frac{(ua \mp vc)z}{uavcz}$$

=the relative excess of b/a over d/c . If u and v are each unity, the process is very simple.

Illustrations: 1. Compare $\frac{5}{8}$ and $\frac{7}{11}$ with $\frac{2}{3}$ as measure.

$$(\frac{5}{8} \div \frac{2}{3}) - 1 = -\frac{1}{8} \text{ and } (\frac{7}{11} \div \frac{2}{3}) - 1 = -\frac{1}{2}.$$

$$\frac{22-16}{22 \times 16} = \frac{1}{58\frac{2}{3}}, \text{ or } = \frac{11-8}{11 \times 8 \times 2} = \frac{1}{58\frac{2}{3}}.$$

$$\text{True answer} = \frac{1}{58}.$$

2. Compare $\frac{3}{8}$ with $\frac{5}{9}$. Take base = $\frac{4}{3}$.

$$\frac{by}{az} = \frac{116}{117}, \quad \frac{dy}{cz} = \frac{196}{195}, \quad \frac{195+117}{195 \times 117} = \frac{312}{22815} = \frac{1}{73.1+}.$$

$$\text{By the formula } \frac{49+29}{49 \times 29 \times 4} = \frac{76}{5684} = \frac{1}{74.8}.$$

$$\text{True answer} = \frac{1911-1885}{1911} = \frac{1}{73.5}.$$

New York, 1878—1883.